ON HEREDITARY RADICALS OF SEMIGROUPS WITH ZERO

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In this note, which can be considered as a continuation of author's paper [14], we shall verify that there exists a common generalization of some assertations of Robert Shulka [11], if we use general hereditary radicals of semigroups with zero, instead of various concrete radicals, e.g. Clifford's [2], Schwarz's [8] radicals, prime and completely prime radicals, respectively, used by R. Shulka. Our generalization also dualizes a result, asserted before for associative rings and alternative rings by author [13], on some join-endomorphisms of the lattice of all twosided ideals of the ring.

The fundamental notions for semigroups can be found in the book's A. H. Clifford and G. B. Preston [3] and E. S. Ljapin [5]. Various concrete radicals for semigroups were discussed e.g. by J. Bosak [1], A. H. Clifford [2], H. J. Hoehnke [4], J. Luh [6], S. Schwarz [8], H. Seidel [9], L. N. Shevrin [10] and author [12]. The possibility to investigate general radicals for semigroups with zero element was shown e.g. by author [14].

For a radical R and for an ideal I of the semigroup S, let $\phi(I)/I$ be the R-radical of S/I.

A radical R is called *hereditary*, if every two sided ideal I of an R-radical semigroup is R-radical.

Proposition 1. The mapping $I\rightarrow\phi(I)$, defined above for ideals of a semigroup, is monotone.

Proof. By D. Rees [7] the first and second isomorphism theorems hold for the factor semigroups. Assume $I_1\subset I_2$ for the ideals $I_j(j=1, 2)$. Then we have:

$$(*)\qquad \phi(I_1)\cup\phi(I_2)/\phi(I_2)\sim\phi(I_1)/\phi(I_1)\cap\phi(I_2).$$

Since here the right side is isomorphic to the factor semigroup $R(S/I_1)/(\phi(I_1)\cap\phi(I_2))/I_1$ it is R-radical. At the same time, the left side of (*) is a two sided ideal of $S/\phi(I_2)\sim(S/I_2)/(\phi(I_2)/I_2)$ hence it is, by Theorem 1 of author [14], also R-semisimple. This yields $\phi(I_1)/\phi(I_1)\cap\phi(I_2)=0$

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consequently the desired inclusion $\phi(I_1)\subset\phi(I_2)$, indeed.

Proposition 2. For arbitrary radical R and two sides ideal I of the semigroup S, if I and S/I are R-radical, then also S is an R-radical semigroup.

Proof. Immediately follows, using the definition of the R-radical (see [14]) and that of Rees factor semigroups and isomorphism theorems (see [7]).

Theorem 3. For an arbitrary hereditary radical R of a semigroup S the mapping $I\to\phi(I)$ is a meet-endomorphism of the lattice of all two sided ideals of S.

Proof. By Proposition 1, we obviously have

$$\phi(I_1)\cap\phi(I_2)\supset\phi(I_1\cap I_2).$$

Since

$$\phi(I_1)\cap\phi(I_2)/\phi(I_1)\cap\phi(I_2)\cap I_j \sim \phi(I_1)\cap\phi(I_2)\cup I_j/I_j\subset\phi(I_j)/I_j$$

and since R is hereditary, $\phi(I_1)\cap\phi(I_2)/\phi(I_1)\cap\phi(I_2)\cap I_j$ is R-radical, for $j=1$ and 2.

Furthermore, e.g. also $\phi(I_1)\cap I_2/I_1\cap I_2$ is R-radical, since it can be isomorphically embedded as an ideal, by the first isomorphism theorem, into $\phi(I_1)/I_1$ and since R is hereditary. Similarly, it can be proved, that $I_1\cap\phi(I_2)/I_1\cap I_2$ is R-radical.

Now, since $I_j\cap\phi(I_k)/I_1\cap I_2$ for $j\neq k$ and $j,k=1,2$ and $\phi(I_1)\cap\phi(I_2)/I_j\cap\phi(I_k)$ are R-radical, by Proposition 2, also $\phi(I_1)\cap\phi(I_2)/I_1\cap I_2$ is R-radical, which implies $\phi(I_1)\cap\phi(I_2)\subset\phi(I_1\cap I_2)$ and hence $\phi(I_1)\cap\phi(I_2)=\phi(I_1\cap I_2)$. Thus $I\to\phi(I)$ is a meet-endomorphism, indeed, which completes the proof.

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